ANALYSIS AND OPTIMIZATION OF AUTO-CORRELATION BASED FREQUENCY OFFSET ESTIMATION

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Abstract: In this letter, a general auto-correlation based frequency offset estimation (FOE) algorithm is analyzed. An approximate closed-form expression for the Mean Square Error (MSE) of the FOE is obtained, and it is proved that, given training symbols of fixed length N, choosing the number of summations in the auto-correlation to be $\langle \frac{N}{3} \rangle$ and the correlation distance to be $\langle \frac{2N}{3} \rangle$ is optimal in that it minimizes the MSE. Simulation results are provided to validate the analysis and optimization.

Key words: Auto-correlation, frequency offset estimation, optimization, performance analysis, un-biased estimator.

1. INTRODUCTION

Carrier Frequency Offset (CFO), caused by frequency deviation between a transmitter and a receiver exists in most communication systems and may result in severe performance degradation or even system failure. Therefore, estimation and compensation of frequency offset in communication systems is important in order to allow coherent demodulation of the transmitted signals. Compared to single-carrier modulation, Orthogonal Frequency Division Multiplexing (OFDM) is more sensitive to frequency offset because it introduces Inter-Carrier Interference (ICI) and destroys the orthogonality among sub-carriers [1]. To mitigate the negative impact of frequency offset, continuous efforts have been made to develop efficient Frequency Offset Estimation (FOE) algorithms.

FOE can be done in the time or frequency domain. In OFDM systems, time-domain algorithms are typically used to estimate the initial frequency offset and frequency-domain algorithms are used to track the residual frequency offset. Time-domain FOE algorithms generally rely on the auto-correlations between two specially designed training signal segments [2–5]. Further enhancements of utilizing training signals composed of multiple identical segments have been proposed in [7, 8]. [9] gives a comparative study of the Schmidl-Cox (SC) [5] and Morelli-Mengali (MM) [6] algorithms for frequency offset estimation in OFDM, along with a new least squares (LS) and a new modified SC algorithm. In [10], the author proposes a novel maximum likelihood (ML) based algorithm for estimating the timing offset and carrier frequency offset in OFDM systems under dispersive fading channels.

Although auto-correlation based FOE algorithms have been used in many practical systems, the performance analysis and optimization of the algorithms has not yet been thoroughly investigated. In this letter, a general auto-correlation based FOE algorithm is analyzed, a closed-form expression for the Mean Square Error (MSE) is derived, and it is proved that if the training symbol length is fixed to be N, to minimize the MSE, the optimal number of summations in the auto-correlation should be $\langle \frac{N}{3} \rangle$ and the optimal auto-correlation distance equals $\langle \frac{2N}{3} \rangle$. This letter is organized as follows: Section 2 introduces a general auto-correlation based frequency offset algorithm. The main result is presented in Section 3. Section 4 presents simulation results and some discussions. Finally, conclusions are drawn in Section 5.

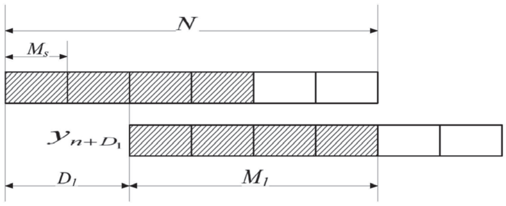

Figure 1: Autocorrelation based FOE

2. AUTO-CORRELATION BASED FREQUENCY OFFSET ESTIMATION

A quasi-static dispersive channel that contains L resolvable multi-paths can be denoted by $\{h_l\}_{l=0}^{L-1}$. Let s_n be the n-th transmitted training symbol with unit energy, then the n-th received symbol can be expressed as

$$y_n = e^{j\theta_n} \sum_{l=0}^{L-1} h_l s_{n-l} + v_n, \tag{1}$$

where v_n is the AWGN with zero mean and variance σ^2 and θ_n is the rotation angle at the n-th symbol caused by the frequency offset. In (1), it is assumed that the rotation angles for L consecutive symbols are approximately the same, this is valid if the frequency offset is not absurdly large.

Let Δf_s be the true frequency offset and T_s be the symbol interval, then θ_n can be expressed as $\theta_n = n\Delta\theta$, where $\Delta\theta$ is the rotation angle per symbol, and is defined as

$$\Delta\theta \triangleq 2\pi T_s \Delta f_s. \tag{2}$$

Auto-correlation based FOE relies on training symbols of length N that are composed of multiple identical segments, each segment has M_s symbols. A sensible design should have $M_s \gg L$.

The auto-correlation metric between y_n and y_{n+D_1} is

$$Q(M_1) = \frac{1}{M_1} \sum_{n=1}^{M_1} (y_n^\dagger)(y_{n+D_1}), \tag{3}$$

where $()^\dagger$ denotes complex conjugation, D_1 is called the"auto-correlation distance", M_1 is the number of summations in the auto-correlation and is called the "complementary auto-correlation distance". Fig.1 illustrates the autocorrelation based FOE, from Fig.1 it is clear that $D_1 = N - M_1$.

Having obtained $Q(M_1)$, the frequency offset can be estimated as [2, 3]

$$\Delta\hat{f}_s = \frac{\angle Q(M_1)}{2\pi D_1 T_s}. \tag{4}$$

If Δf_s is in the range $\left(-\frac{1}{2D_1T_s}, \frac{1}{2D_1T_s}\right)$, equation (4) can provide correct estimation, otherwise there exists a 2π or multiples of 2π phase ambiguity. In this case, the correct rotated angle should be $\angle Q(M_1) + 2\pi d$ instead of $\angle Q(M_1)$, where d is an integer. To resolve the phase ambiguity, another auto-correlation metric with a shorter auto-correlation distance $D_2 \triangleq (N - M_2)$ can be used, i.e., calculating

$$Q(M_2) = \frac{1}{M_2} \sum_{n=1}^{M_2} (y_n^\dagger)(y_{n+D_2}), \tag{5}$$

where M_2 is the corresponding complementary auto-correlation distance. Clearly, the two auto-correlation metrics have the relation

$$\frac{D_1}{D_2} \angle Q(M_2) \approx \angle Q(M_1) + 2\pi d, \tag{6}$$

and the $2\pi d$ phase ambiguity can be estimated as

$$\hat{d} = \left\langle \frac{\frac{D_1}{D_2}\angle Q(M_2) - \angle Q(M_1)}{2\pi} \right\rangle, \tag{7}$$

where $\langle\cdot\rangle$ is the rounding operation. Then, the estimated frequency offset equals

$$\Delta\hat{f}_s = \frac{\angle Q(M_1) + 2\pi\hat{d}}{2\pi D_1 T_s}. \tag{8}$$

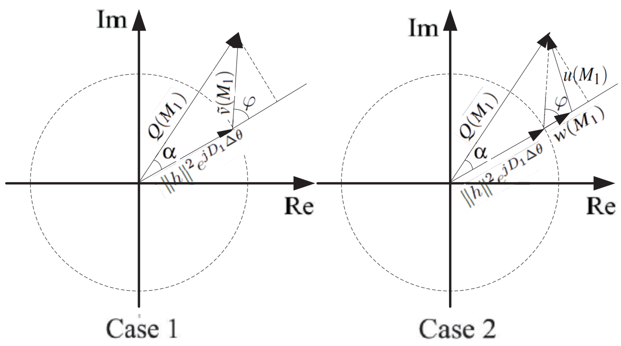

Figure 2: Illustration of angle approximation induced by $\tilde{v}(M_1)$

In the autocorrelation based FOE algorithm introduced above, the FOE precision is mainly determined by M_1 and the range of resolved frequency offset is determined by M_2. In the following, we analyze the performance of the auto-correlation based FOE algorithm, and show how to optimize the algorithm.

3. PERFORMANCE ANALYSIS AND PARAMETER OPTIMIZATION

For the auto-correlation based FOE algorithm, clearly, the larger the auto-correlation distance (i.e., D_1 or D_2) is, the finer the estimated frequency offset, and the better the performance. However, given a fixed training symbol length N, large auto-correlation distances mean smaller complementary auto-correlation distances (i.e. M_1 or M_2). The smaller the complementary auto-correlation, the lesser the number of samples used to calculate the auto-correlation metric and thus leading to poor performance. Therefore, given N, there is an optimal auto-correlation distance where the MSE is minimized.

Since M_2 is only used to resolve the ambiguity, it is sufficient to choose M_2 to satisfy the following inequality

$$-\pi < 2\pi(N - M_2)\Delta f_s T_s < \pi. \tag{9}$$

In the following, we only focus on how to optimize the parameter M_1. We first derive the MSE of the estimated frequency offset with complementary auto-correlation distance M_1.

Because of the repeated segments, D_1 is a multiple of M_s, and y_{n+D_1} equals

$$\begin{aligned} y_{n+D_1} &= e^{j(n+D_1)\Delta\theta}\sum_{l=0}^{L-1} h_l s_{n+D_1-l} + v_{n+D_1} \\ &= e^{j(n+D_1)\Delta\theta}\sum_{l=0}^{L-1} h_l s_{n-l} + v_{n+D_1}. \end{aligned} \tag{10}$$

Let us define $z_n \triangleq \sum_{l=0}^{L-1} h_l s_{n-l}$. Assuming independent and unit energy training symbols s_n, we have $\mathbb{E}\left[\|z_n\|^2\right] = \|h\|^2 \triangleq \sum_{l=0}^{L-1}\|h_l\|^2$, and as M_1 gets large, we have the following approximation

$$\frac{1}{M_1}\sum_{n=1}^{M_1}\|z_n\|^2 \approx \|h\|^2. \tag{11}$$

Substituting (10) into (5) and using the above approximation, $Q(M_1)$ can be expressed as

$$\begin{aligned} Q(M_1) &= \frac{1}{M_1}\sum_{n=1}^{M_1}\|z_n\|^2 e^{jD_1\Delta\theta} + \tilde{v}(M_1) \\ &\approx \|h\|^2 e^{jD_1\Delta\theta} + \tilde{v}(M_1), \end{aligned} \tag{12}$$

where $\tilde{v}(M_1)$ is called the "noise term" for FOE and is given by

$$\tilde{v}(M_1) = A + B + C, \tag{13}$$

where A, B and C are defined as:

$$A \triangleq \frac{1}{M_1}\sum_{n=1}^{M_1}\left(v_n^\dagger z_n e^{j(n+D_1)\Delta\theta}\right), \tag{14}$$

$$B \triangleq \frac{1}{M_1}\sum_{n=1}^{M_1}\left(v_{n+D_1} z_n^\dagger e^{-jn\Delta\theta}\right), \tag{15}$$

$$C \triangleq \frac{1}{M_1}\sum_{n=1}^{M_1}\left(v_n^\dagger v_{n+D_1}\right). \tag{16}$$

Using equation (12) and resolving the $2\pi d$ ambiguity, we obtain

$$\angle Q(M_1) + 2\pi d = D_1\Delta\theta + \alpha, \tag{17}$$

where α is the angle induced by noise term $\tilde{v}(M_1)$. Note that $\alpha \neq \angle\tilde{v}(M_1)$, instead, it is the angle between $Q(M_1)$ and $e^{jD_1\Delta\theta}$ (See Fig.2).

The estimation of Δf_s in equation (8) can be derived as

$$\Delta\hat{f}_s = \Delta f_s + \frac{\alpha}{2\pi D_1 T_s}. \tag{18}$$

$\Delta\hat{f}_s$ is later shown to be an unbiased estimator, and the MSE of the estimated frequency offset is given by

$$R \triangleq \frac{\mathbb{E}\left[\|\alpha\|^2\right]}{4\pi^2 D_1^2 T_s^2}. \tag{19}$$

To get optimal FOE performance, M_1 should be chosen to satisfy

$$M_1^{opt} = \arg\min_{M_1}\{R\}. \tag{20}$$

The following theorem summarizes the main result of this letter, which gives M_1^{opt}, and the minimum MSE.

Theorem *1: For a system with N training symbols for FOE, the optimal complementary auto-correlation distance that minimizes the MSE of the estimated frequency offset is*

$$M_1^{opt} = \left\langle \frac{N}{3} \right\rangle$$

and the corresponding minimum MSE is approximately

$$R^{min} \approx \frac{1}{8\pi^2 T_s^2\left(N - \left\langle\frac{N}{3}\right\rangle\right)\left\langle\frac{N}{3}\right\rangle^2}\left(\frac{2}{SNR} + \frac{1}{SNR^2}\right),$$

where $SNR \triangleq \frac{\|h\|^2}{\sigma^2}$.

Proof: Expanding the expectation of $\|\tilde{v}(M_1)\|^2$ in (13), we have

$$\begin{aligned} \mathbb{E}\left[\|\tilde{v}(M_1)\|^2\right] &= \mathbb{E}\left[\|A+B\|^2\right] + \mathbb{E}\left[(A+B)C^\dagger\right] \\ &\quad + \mathbb{E}\left[C(A+B)^\dagger\right] + \mathbb{E}\left[\|C\|^2\right]. \end{aligned}$$

Since v_n is a complex Gaussian random variable with zero mean, we have $\mathbb{E}\left[(A+B)C^\dagger\right] = 0$ and $\mathbb{E}\left[C(A+B)^\dagger\right] = 0$. Therefore, $\mathbb{E}\left[\|\tilde{v}(M_1)\|^2\right]$ can be simplified to

$$\mathbb{E}\left[\|\tilde{v}(M_1)\|^2\right] = \mathbb{E}\left[\|A+B\|^2\right] + \frac{\sigma^4}{M_1}. \tag{21}$$

Case 1: $M_1 \leq \left\langle\frac{N-1}{2}\right\rangle$

In this case, there is no overlap between v_n and v_{n+D_1} for $n = 1, 2, \cdots, M_1$, so A and B are independent zero mean circular complex Gaussian random variables. Since $\tilde{v}(M_1)$ does not favor any specific direction, we have $\mathbb{E}[\alpha] = 0$. This makes $\Delta\hat{f}_s$ given in equation (18) an unbiased estimator.*

Based on the illustration in Fig.2, assuming M_1 is large, in high SNR scenarios, the angle α can be approximated as

$$\alpha \approx \frac{\|\tilde{v}(M_1)\|\sin\varphi}{\|h\|^2}, \tag{22}$$

where φ is the angle between $\tilde{v}(M_1)$ and $e^{jD_1\Delta\theta}$. In this

*It is important to note that the distribution of α in equation(18) is unknown even though the first and second moments are known. Since the distribution is unknown, the CRLB cannot be derived for this dedicated case.

case, R can be approximated as

$$R \approx \frac{\mathbb{E}\left(\|\tilde{v}(M_1)\|^2\right) - \mathbb{E}\left(\cos 2\varphi \|\tilde{v}(M_1)\|^2\right)}{8\pi^2 D_1^2 T_s^2 \|h\|^4}.$$

We also have

$$\mathbb{E}\left[\cos 2\varphi \|\tilde{v}(M_1)\|^2\right] = 0, \tag{23}$$

where we have applied the property that φ is uniformly distributed and independent to the length of $\|\tilde{v}(M_1)\|$.

The expectation $\mathbb{E}\left[\|\tilde{v}(M_1)\|^2\right]$ equals

$$\begin{aligned} \mathbb{E}\left[\|\tilde{v}(M_1)\|^2\right] &= \mathbb{E}\left[\|A\|^2\right] + \mathbb{E}\left[\|B\|^2\right] + \frac{\sigma^4}{M_1} \\ &\approx \frac{2\|h\|^2\sigma^2 + \sigma^4}{M_1}. \end{aligned} \tag{24}$$

Using the relation $D_1 = N - M_1$ and combining equations (23) and (24), R becomes

$$\begin{aligned} R_1 &\approx \frac{2\|h\|^2\sigma^2 + \sigma^4}{8\pi^2 T_s^2 \|h\|^4 M_1 (N - M_1)^2} \\ &= \frac{1}{8\pi^2 T_s^2 M_1 (N - M_1)^2} \left(\frac{2}{SNR} + \frac{1}{SNR^2}\right). \end{aligned} \tag{25}$$

The optimization problem (20) is now equivalent to

$$M_1^{opt} = \arg\max_{M_1} \left\{M_1 (N - M_1)^2\right\}. \tag{26}$$

It is not difficult to show that

$$M_1^{opt} = \left\langle \frac{N}{3} \right\rangle, \tag{27}$$

and the corresponding minimum MSE is

$$R^{min} = \frac{\frac{2}{SNR} + \frac{1}{SNR^2}}{8\pi^2 T_s^2 \left(N - \left\langle \frac{N}{3}\right\rangle\right) \left\langle \frac{N}{3} \right\rangle^2}. \tag{28}$$

Case 2: $M_1 > \left\langle \frac{N-1}{2} \right\rangle$

In this case, A and B are NOT independent anymore because the $(k+D_1)$-th term in A, which is

$$A_{k+D_1} = \frac{v_{k+D_1}^{\dagger} z_{k+D_1} e^{j(k+D_1)\Delta\theta} e^{jD_1\Delta\theta}}{M_1}, \tag{29}$$

and the k-th term in B, which is

$$B_k = \frac{v_{k+D_1} z_k^{\dagger} e^{-jk\Delta\theta}}{M_1} = \frac{v_{k+D_1} z_{k+D_1}^{\dagger} e^{-jk\Delta\theta}}{M_1}, \tag{30}$$

are correlated, and the terms $A_{k+D_1} + B_k$ for $k = 1, 2, \cdots, (M_1 - D_1)$ are along the same direction as $e^{jD_1\Delta\theta}$, because $A_{k+D_1} + B_k$ can be written as

$$A_{k+D_1} + B_k = \frac{2\Re\left\{v_{k+D_1}^{\dagger} z_{k+D_1} e^{j(k+D_1)\Delta\theta}\right\} e^{jD_1\Delta\theta}}{M_1}.$$

Regrouping the terms in $A+B$, we obtain

$$A + B = \sum_{n=1}^{D_1} (A_n + B_{n+M_1-D_1}) + \underbrace{\sum_{k=D_1+1}^{M_1} A_k + B_{k-D_1}}_{\triangleq w(M_1)},$$

where $w(M_1)$ is the summation of correlated terms and is along the direction of $e^{jD_1\Delta\theta}$, so it has no contribution to the angle α. Then, $\tilde{v}(M_1)$ can be re-written as

$$\tilde{v}(M_1) = \underbrace{\left(\sum_{n=1}^{D_1} (A_n + B_{n+M_1-D_1}) + C\right)}_{\triangleq u(M_1)} + w(M_1). \tag{31}$$

Using similar arguments as in Case 1, we have $\mathbb{E}[\alpha] = 0$, which leads to an unbiased estimation of $\Delta\hat{f}_s$ given by equation (18).

Based on the illustration in Fig.2, we can approximate the angle α as

$$\alpha \approx \frac{\|u(M_1)\| \sin\varphi}{\|h\|^2}, \tag{32}$$

where φ is the angle between $u(M_1)$ and $e^{jD_1\Delta\theta}$.

Following the same procedure as in Case 1, we have

$$\mathbb{E}\left[\|u(M_1)\|^2\right] \approx \frac{2(N - M_1)\|h\|^2\sigma^2}{M_1^2} + \frac{\sigma^4}{M_1}, \tag{33}$$

and the corresponding MSE equals

$$R_2 \approx \left(\frac{\frac{2}{SNR}}{8\pi^2 T_s^2 M_1^2 (N - M_1)} + \frac{\frac{1}{SNR^2}}{8\pi^2 T_s^2 M_1 (N - M_1)^2}\right). \tag{34}$$

Define $L(M_1) \triangleq \frac{1}{8\pi^2 T_s^2 M_1 (N-M_1)^2 SNR}$ and $D \triangleq L(M_1^{opt})$, where M_1^{opt} is given by equation (27). R_1 and R_2 given by equations (25) and (34), respectively can then be re-written as

$$\begin{aligned} R_1(M_1) &= L(M_1)\left(2 + \frac{1}{SNR}\right) \\ R_2(M_1) &= L(M_1)\left(\frac{2N}{M_1} - 2 + \frac{1}{SNR}\right) \end{aligned}$$

We know that $L(M_1) \geq D$, and the minimum value of R_1 is $R_1^{min} = D(2 + \frac{1}{SNR})$. To complete the proof we show that

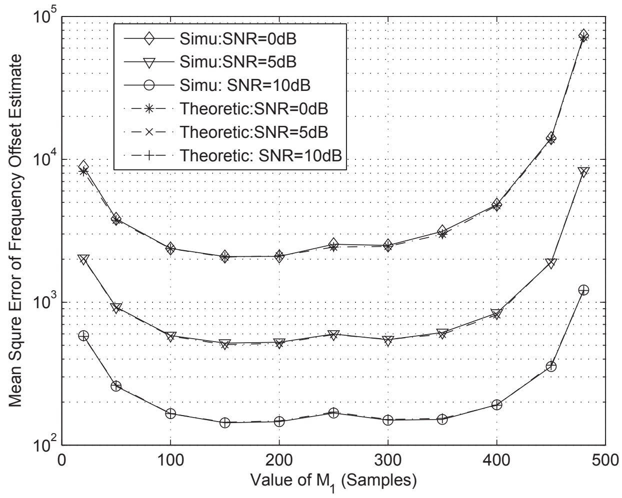

Figure 3: Validation of approximated analysis and parameter optimization.

$R_2(M_1) > R_1(M_1^{opt}) = D\left(2+\frac{1}{SNR}\right)$.

$$L(M_1) \geq D \tag{35}$$

$$R_2(M_1) \geq D\left(\frac{2N}{M1} - 2 + \frac{1}{SNR}\right) \tag{36}$$

$$\tag{37}$$

R_2 is bounded by

$$R_2(M_1) > D\left(\frac{2N}{(N/2)} - 2 + \frac{1}{SNR}\right) \tag{38}$$

$$R_2(M_1) > D(2+A) = R_1^{min} \tag{39}$$

Therefore, the minimum MSE in Case 1 is a global minimum. ■

4. SIMULATION VALIDATIONS AND DISCUSSION

To validate the analysis and optimization, we consider a communication system that has $N = 500$ symbols, $\Delta f_s =$ 10kHz, and $1/T_s =$ 1MHz. To satisfy (9), we choose $M_2 =$ 480 symbols.

The simulated and theoretical results of MSE vs. M_1 are shown in Fig.3. It can be seen that the MSE calculated from our analysis matches the simulated MSE very well, and the minimum MSE is achieved when $M_1 = 167 = \left\langle \frac{500}{3} \right\rangle$, as predicted by Theorem 1.

From Fig.3, it can be observed that the curve for $SNR =$ 10dB is more symmetric than the curve for $SNR =$ 0dB and the local minimum in the curve of $SNR =$ 10dB is closer to the global minimum. This is because at high SNRs, the $\frac{1}{SNR^2}$ term in (25) and (34) can be ignored and the MSE becomes $R \approx \frac{2}{8\pi^2 T_s^2 M_1 (N-M_1)^2 (SNR)}$ and $R \approx \frac{2}{8\pi^2 T_s^2 M_1^2 (N-M_1)(SNR)}$, for Case 1 and Case 2, respectively. They are symmetric to the center $M_1 = \left\langle \frac{N-1}{2} \right\rangle$ and reach the same minimum when $M_1 = \left\langle \frac{N}{3} \right\rangle$ and $M_1 = N - \left\langle \frac{N}{3} \right\rangle$, respectively.

As a last comment, from the closed-form MSE formulas, we can see that, when N is fixed, the MSE of FOE is just a function of M_1 and SNR, and is independent of Δf_s.

5. CONCLUSION

In this letter, a general auto-correlation based FOE algorithm was analyzed, closed-form expressions of the MSE were derived, and it was proved that the optimal complementary auto-correlation distance equals $\left\langle \frac{N}{3} \right\rangle$, where N is the total number of training symbols. The results obtained in the letter can be of practical usage when designing training symbols in the implementation of auto-correlation based FOE algorithms.

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